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Getting numbers into and back from locally convex topology:
Part 1: (weak) locally convex approach structures

Approach theory as started by R.Lowen in the mid 80's, draws heavily upon the otivation of providing a remedy for the lack of productivity of the concept of metrizability in topology: an arbitrary uncountable product of metrizable topological spaces is hardly ever metrizable, and for countably infinite products, there is no preferred, 'canonical' metric on the product which does the job. However, suitably axiomatized point-set distances which cannot be retrieved from point-point distances (as in the metric case) seem to capture exactly that part of the metric information that can be retained in concordance with topological products. The topological construct **Ap** of approach spaces and contractions yields a category containing both the category of topological spaces and continuous maps **Top**, and the category of metric spaces and non-expansive maps Met as (full !) subcategories. An interesting phenomenon occurs when one tries to 'merge' these approach structures with algebraic operations like for example vector space operations (in analogy to topological vector spaces): simply internalizing the vector space operations in **Ap** yields a totally unsatisfactory category, since the object **R** of the reals with its canonical Euclidean norm does not belong to it, for the addition is a 2-Lipschitz map and the multiplication not even Lipschitz. The category **ApVec** which we will introduce, solves the productivity problem mentioned above for vector pseudometrics. This constitutes an acceptable candidate deserving the name of a 'quantification' of **TopVec**, the category of topological vector spaces, since it contains both **TopVec** and the category of all vector-pseudometric spaces as full subcategories in a neat way. The situation becomes even nicer: by looking at the epireflective hull of Norm (the category of seminormed spaces and linear non-expansive maps) in **ApVec**, the category of locally convex approach spaces lc**ApVec** is obtained . This is solving again the productivity problem from above, now for seminorms in relation to locally convex topological spaces. Throughout the paper we denote the category of locally convex topological spaces by lc**TopVec**. It is working in **ApVec** and lc**ApVec** which we propose as 'quantified functional analysis'. Classically for (non-metrizable/non-semi-normable) topological vector spaces, only an 'isomorphic' theory exists, whereas now, when we start e.g. from a well-chosen defining set of vector pseudometrics/seminorms, working on the approach level now allows an 'isometric' theory, where canonical numerical concepts exists. For example, one can express 'how far is a vector away from being a limit point of a given net', rather than just knowing whether or not the filter or net converges to the vector. We will discuss how one can capture weak structures in this setting as a guiding example.

[2000 MSC: 18B99, 18D15, 46A03, 46B04, 46M05, 46M16]